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On kinetic Ising models in one dimension

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Abstract. It is shown that Glauber dynamics in 1D Ising spin systems is not universal. This is illustrated on a periodic model with a basic unit cell $\{J_1, \ldots, J_n\}$ containing an arbitrary set of n ferromagnetic coupling constants. The dynamic critical exponent z is calculated exactly as $z = 1 + \max\{J_i\}/\min\{J_i\}$, the known value $z = 2$ is recovered only for $n = 1$. The extension of this result to other types of dynamics is briefly discussed.

1. Introduction

Kinetic Ising models correspond to the simplest dynamic pictures of real systems and are believed to provide a sensible description of the dynamics of a large number of physical systems [1]. As far as analytic results are concerned, only the pure Ising chain with different spin-flip dynamics has been completely solved within the framework of Glauber dynamics [2]. In particular, the critical dynamic exponent z for various general types of single-spin-flip dynamics has been determined rigorously in some cases. Deviations [3] from the dynamic scaling [4] value $z = 2$ have been obtained and these results were confirmed subsequently [5] using simple physical arguments. In this respect, the occurrence of different dynamic universality classes is closely associated with different types of dynamics. In this paper we address the universality question from another point of view originating from the fact that the static critical behaviour of 1D Ising systems is controlled by a zero-temperature fixed point. For this we have investigated the standard Glauber dynamics on a simple periodic model, in which the basic cell contains different coupling constants $J_1, J_2, \ldots, J_n$ where n is an integer. The critical dynamic exponent z associated with this regular structure can actually be calculated exactly for general n. It turns out that the precise value of z is a non-trivial combination of J_i implying in particular its non-universality. The physical origin of this peculiar result can, however, be traced back to the low-T dynamics of defects (kinks). It is important to notice that a similar situation arises in the spin dynamics on percolation clusters at threshold [6, 7] and some fractal lattices where a more singular dynamics takes place with $z \simeq J/T$, i.e. a temperature-dependent exponent. The common feature in these two cases is actually the same and leads to a violation of dynamic scaling. In fact, in both these cases the static critical behaviour is governed by a zero-temperature fixed point. The same holds for the dynamics which is actually dominated by the motion of thermally activated defects and then strongly dependent on topology and interaction strengths.

This paper is organised as follows. The model is introduced in § 2 where some equilibrium properties are briefly presented. An explicit expression is obtained for the

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critical dynamic exponent z in this section. In § 3 the domain boundary diffusion arguments of [5] are extended to the present model, yielding the exact result for z. Finally, § 4 contains a discussion of the results with some comments on possible generalisations to other types of single-spin-flip dynamics.

2. Glauber dynamics of the periodic model

The periodic 1D Ising model studied in this section is defined by the Ising Hamiltonian

$$\mathcal{H} = -\sum_{i=1}^{N} J_i \sigma_i \sigma_{i+1} \qquad \sigma_i = \pm 1 (\sigma_{N+1} = \sigma_1).$$

The interaction strengths between the nearest-neighbour spins have the values $J_1, J_2, \ldots, J_n$ and are periodically distributed with a basic cell $[J_1, J_2, \ldots, J_n]$. This system can be regarded as a linear array of identical Ising spin 'molecules' which consist of n Ising spins, with intramolecular interaction strengths $J_1, J_2, \ldots, J_{n-1}$ and intermolecular interaction strength J_n, i.e. $J_i = J_q$ for $1 \leqslant i = pn + q \leqslant n$.

The static critical properties of the above model are trivially obtained from the exact expression of the partition function. Here we quote just two results which are of some interest for the spin dynamics. The average spin values at all sites vanish: $\langle \sigma_i \rangle = 0$ and the spin correlations are $\langle \sigma_i \sigma_{i+1} \rangle = \tanh K_i$, $K_i \equiv J_i / T$. Similarly the thermal spin-spin correlation length $\xi(T)$ diverges near the critical temperature $T_c = 0$ as

$$\xi(T) \simeq \exp(2K_m) \tag{1}$$

where

$$K_m = \min_{1 \leqslant i \leqslant n} (K_i).$$

Since the above interacting spin system has no intrinsic dynamics, a stochastic approach is usually used in order to mimic the interaction with the heat bath. In the single-spin-flip dynamics initiated for the Ising chain by Glauber, all the information about the chain at time t is contained in the probability function $P(\{\sigma_i\}, t)$ which is the probability that the spin configuration of the chain is $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ at time t. The time evolution of $P(\{\sigma_i\}, t)$ is given by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\{\sigma_i\}, t) = \sum_j \omega_j(\sigma_1, \ldots, -\sigma_j, \ldots, \sigma_N) P(\sigma_1; \sigma_2, \ldots, -\sigma_j, \ldots, \sigma_N, t)$$

$$- \sum_j \omega_j(\sigma_1, \ldots, \sigma_j, \ldots, \sigma_N) P(\sigma_1, \sigma_2, \ldots, \sigma_j, \ldots, \sigma_N, t). \tag{2}$$

In (2), $\omega_j(\sigma_j)$ is the transition rate for the process $\{\sigma_1, \ldots, \sigma_j, \ldots, \sigma_N) \rightarrow \{\sigma_1, \ldots, -\sigma_j, \ldots, \sigma_N\}$. For a linear chain with nearest-neighbour interactions, the spin dependence of $\omega_j(\sigma_j)$ is assumed to be a function of the local field h_j at site j. The most general form satisfying the requirement of detailed balance and linear dependence on σ_j is the following:

$$\omega_j(\sigma_i) = \tau_0^{-1} [1 - \sigma_j \tanh(h_j / T)] \tag{3}$$

where $h_j = \Sigma_i J_{ij} \sigma_i$ is the local field.

The explicit expression of (3) can be written as

$$\omega_j(\sigma_j) = \tau_0^{-1}[1 - \tfrac{1}{2}\sigma_j(\gamma_j^+\sigma_{j-1} + \gamma_j^-\sigma_{j+1})] \tag{4}$$

with the notations

$$\gamma_j^{\pm} = \tanh(K_{j-1} + K_j) \pm \tanh(K_{j-1} - K_j). \tag{5}$$

Equations (2.5) allow in principle for a complete solution of the Glauber dynamics. Here we simply consider the magnetisation relaxation during time

$$q_j(t) \equiv \sum_{\{\alpha\}} \sigma_j P(\{\sigma_j\}, t) \qquad 1 \leqslant j \leqslant N \tag{6}$$

which, using (1), is a solution of the linear system ($\tau_0 = 1$):

$$\frac{\mathrm{d}}{\mathrm{d}t} q_j(t) = -2\langle \sigma_j(t)\omega_j(\sigma_j)\rangle$$

$$= -q_j(t) + \tfrac{1}{2}(\gamma_j^- q_j(t) + \gamma_j^+ q_{j+1}(t)). \tag{7}$$

Next we specialise (7) to the present model. For this we define the sublattice magnetisation

$$m_j = \sum_{i=0}^{k-1} \sigma_{j+in} \qquad 1 \leqslant j \leqslant n \qquad k \equiv N/n$$

being the number of basic cells. Using (7) one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} m_1(t) \\ \vdots \\ m_n(t) \end{bmatrix} = M \begin{bmatrix} m_1(t) \\ \vdots \\ m_n(t) \end{bmatrix} \tag{8}$$

where M denotes a $n \times n$ matrix:

$$M = \begin{bmatrix} -1 & \gamma_1^- & 0 & \dots & 0 & \gamma_1^+ \\ \gamma_2^+ & -1 & \gamma_2^- & & & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \gamma_{n-1}^- \\ \gamma_n^- & 0 & 0 & & \gamma_n^+ & -1 \end{bmatrix}. \tag{9}$$

The relaxation of $\{m_j(t)\}$ towards the equilibrium is governed by the set $\tau_i = -1/\lambda_i$ of n relaxation times, where $\{\lambda_i\}$ are the eigenvalues of matrix M. The asymptotic long-time behaviour of $\{m_j\}$ is given by the lowest eigenvalue. Explicit calculation of this spectrum can only be achieved for small values of n. For instance, the case $n = 2$ has been worked out in [8] and subsequently in [9] and this for modelling the kinetics of an alternating copolymer. However, at very low temperature there is no need for the full spectrum $\{\tau_i\}$, the main result can be deduced from the following observation. For generic $\{J_1, \dots, J_n\}$, the eigenvalues λ_i are real, non-degenerate and the longest relaxation time τ_M dominates the sum $\Sigma\, \tau_i$ at low T. Therefore the relaxation time τ_M dominates the sum $\Sigma_{i=1}^n \tau_i$ at low T. Therefore the sufficiently low temperature, the relevant relaxation time is given by

$$\tau_M \equiv \max\, \tau_i \simeq -\sum_{i=1}^{n} \lambda_i^{-1} \tag{10}$$

and this corresponds to the dominant relaxation rate. Accordingly the problem is reduced to the calculation of the sum $\Sigma_{i=1}^{n} \lambda_i^{-1}$ which in turn can be expressed as the ratio A_1/A_0 of the first two coefficients of the characteristic polynomial

$$P(\lambda) \equiv \sum_{i=1}^{n} A_i \lambda^i$$

of the matrix M.

A convenient way for the calculation of $P(\lambda)$ is provided by a transfer-matrix formulation of (8) and this leads to

$$\sum_{i=1}^{n} \tau_i = \mathcal{N}/\mathcal{D}.$$

Here $\mathcal{N}$ and $\mathcal{D}$ are defined as follows. $\mathcal{N}$ is the coefficient of λ in the polynomial

$$\mathrm{Tr} \prod_{i=n}^{1} M_i(\lambda)$$

and $\mathcal{D}$ is the constant coefficient in the sum:

$$\mathrm{Tr} \prod_{i=n}^{1} M_i(\lambda) - 2 \prod_{i=1}^{n} b_i.$$

In these expressions we have used the following notations:

$$M_i(\lambda) = \begin{pmatrix} (1+\lambda)a_i & c_i \\ b_i & 0 \end{pmatrix} \tag{11}$$

where

$$a_i = (x_i + x_{i+1} - x_i x_{i+1})(2 - x_i - x_{i+1} + x_i x_{i+1})$$

$$b_i = x_i(2 - x_i)(1 - x_{i+1})$$

$$c_i = -x_{i+1}(2 - x_{i+1})(1 - x_i)$$

and

$$x_i = 1 - \tanh K_i.$$

Since we are only interested in the low-temperature limit, the above formulation allows for a simple expansion of $\mathcal{N}$ and $\mathcal{D}$ in powers of x_i. For this, it is convenient to expand the product

$$\prod_{i=n}^{1} M_i(\lambda)$$

on the algebra generated by the matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

A straightforward but lengthy calculation leads to the simple expressions, valid at $x_i \ll 1$:

$$\begin{aligned} &\mathcal{D}/2^n \simeq PS^2 \\ &\mathcal{N}/2^{n+1} \simeq \sum_{i=1}^{n} P(S - x_i)/x_i + nP \end{aligned} \tag{12}$$

where

$$P \equiv \prod_{i=1}^{n} x_i \qquad \text{and} \qquad S \equiv \sum_{i=1}^{n} x_i.$$

When used in (10), this result leads to a simple expression for τ_M:

$$\tau_M \simeq \exp(2K_m + 2K_M) \tag{13}$$

with $K_m = \min_i K_i$ and $K_M = \max_i K_i$ respectively.

This implies in particular the announced value for the critical dynamic exponent z ($\tau_M \sim \xi^z(T)$):

$$z = 1 + \max\{J_i\}/\min\{J_i\}. \tag{14}$$

This non-trivial expression for z calls for some remarks. As expected, (14) reproduces the known result $z = 2$ for $J_1 = J_2 = \ldots = J_n$ but in general z is enhanced in comparison with the pure chain result. This increase of z, without disorder, has already been pointed out in the simple case $n = 2$ [9]. In this respect (14) is the natural extension of this result for arbitrary n. Furthermore, in contrast with the static critical exponents which are independent of $\{J_i\}$, the exponent z assumes a non-universal value. As will be seen in the next section, similar expressions are expected to hold for other kinetic models such as Kawasaki dynamics [10] as well as Potts spin systems.

3. Physical origin of non-universality

The Ising chain in equilibrium can be viewed as being composed of domains of parallel spins. Kinks or domain boundaries are simply bonds joining two such domains of opposite magnetisations. The probability of finding a kink at the ith bond is given by $\frac{1}{2}(1 - \langle \sigma_i \sigma_{1+i} \rangle) \simeq e^{-2K_i}$ and the most probable kink is associated with the smallest K_i. Therefore at low T, the domains are of average length $\xi(T)$ and the magnetisation relaxation is governed by the motion of kinks. For instance, in the pure chain the dominant process by which the domains decay in time is by diffusion of the kinks and this implies $\tau \sim \xi^2$, i.e. $z = 2$ [5]. This argument must be modified in the case discussed here and (13) can actually be recovered as described below.

Due to the presence of different values of bond strengths, the motion of kinks is not a simple random walk. The transition rates to the left and to the right depend, of course, on the location of the kink, but in general they are not equal. At low enough temperature, the timescale of the kink motion is then $\tau \sim \xi(T)/R \sim \xi^z(T)$ where R is the rate of motion. For sufficiently low T, R is governed by the largest energy barrier, i.e. the strongest bond. This rate is therefore given by $R \sim \exp(-2K_M)$ and (13) follows immediately.

This argument, reproducing (13) in simple terms, shows clearly the origin of the non-universal value of the exponent z. The precise value of z involves a length scale $\xi(T)$ and a timescale $\tau(T)$. In the present case $\xi(T)$ is controlled by the smallest interaction K_m whereas $\tau(T)$ is fixed by the largest energy barrier which turns out to be dominated by the strongest bond K_M. These features are implied by the 1D topology of the lattice and the discrete nature of the dynamic degrees of freedom. In this respect, it is not difficult to extend the above argument to other situations sharing the same features as the model calculated in § 2.

4. Discussion

As pointed out in § 1, the peculiar result of this paper is a direct consequence of the zero-temperature fixed point. This feature of the Ising chain is already visible in the renormalisation group analysis of the pure case [11]. Universal values of z are, however, expected to appear in 2D lattices such as the checkerboard Ising model where T_c is known exactly for arbitrary coupling strengths [12]. We conclude by noting that more interesting situations arise in disordered systems (spin glasses, random field Ising models, charge density waves, etc) where zero-temperature fixed points can drive the dynamics of these systems. In this respect we mention the recent investigation [13] of the glassy dynamics of charge density wave systems where both the relaxation functions and dynamic exponents are fixed by the probability distribution of pinning strengths.

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